On the Usefulness of Rigor in Teaching Mathematics as Illustrated with Absolute Value

by Zero Richardson

I have definitely come full circle in my opinions of mathematical rigor.

Don't get me wrong, I've always seen the point of rigor for students pursuing mathematics and especially advanced mathematics, but I was almost never sold on its purpose in a liberal arts curriculum or even those degrees/diplomas requiring nothing more than a couple levels of calculus, let alone for a high school student.

Thankfully, I've grown up some since then and I'm now somewhere in the middle of the debate. Let's go through varying levels of rigor as it applies to absolute value to see what I mean.

Consider the absolute value of a number. If you present the words and ask students to give a definition, they may state something along the lines of "the positive of a number" or "the distance a number is from zero".

The second is what I refer to as a "heuristic definition" of absolute value. It has real pedagogical worth and even when students forget what they've memorized concerning formulas and rules, they can usually recall the heuristic definition.

I adore heuristic definitions and I think they should be utilized whenever possible, but they do have their limitations.

More on limitations of the heuristic definition:

For instance, if absolute value is distance from zero, then what is the absolute value of the point $(-3,4)$?

If it's distance from zero, we should be able to use absolute value for this right? We can definitely find the distance $(-3,4)$ is from the origin: just put a ruler on the graph . . . but what does this have to do with the absolute value of $(-3,4)$? Can we even talk about the absolute value of a number defined this way or are we finding something else?

This leads much further down the rabbit hole of mathematics than I'd like to go into just yet, but it's a great way to start a discussion on metrics and how we define distance in general.

The next obvious discussion after defining a 2D distance is to ask, "Now what if we overlaid a grid on a map of a city and wanted to know how far it is to get from school to a restaurant. Can we use the same distance as before?"

So what about the "positive of a number" definition? Well, for starters, it's incorrect. Even if we have some idea of what the "positive of a number" means, it's promoting the idea that the absolute value of something is always positive—it's not by the way—and this leads to one of the most commonly missed true/false questions in algebra courses: zero is neither positive nor negative.

The “positive of a number” is not a heuristic definition nor is it a definition of any variety nor is it something that should ever be uttered in a classroom. It has negative pedagogical value. That is, it will actively harm the education a student is getting.

There’s nothing wrong with “distance a number is from zero” however and it even dovetails nicely with the advanced subject of metrics—a subject in which absolute value has a special place and is usually the first example therein.

However, it is rather limiting, and it is also circular. First of all, as anyone that has studied metrics knows, the idea of “distance” is defined with metrics, not something that defines metrics. You shouldn’t call some operator M a metric because it’s a distance without an understanding of how that distance is calculated or what a distance is.

So first we ask: Does anyone know how absolute value of a number calculates distance?

Your students will probably respond with something like, whenever there’s a negative, it becomes a positive and it’s that many units away from zero.

Secondly, we ask something like: Can you use “distance from zero” to solve the inequality $|x| > 2$? Or the inequality $|x| < 2$? Or even the equation $|x| = 2$?

For most students, the answer is not really. And asking your students to do so is both obscuring true mathematics and further perpetuating the idea that to understand math you just have to magically know it. Some students will be able to intuitively understand the solutions to these equations based on their innate understanding of absolute value—and that is great for those students—but it doesn’t teach anything to them or the rest of the class.

> Note: Even for those students that do intuitively understand it at *these* levels , there are more levels to come.
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> For instance, having operations in and around the absolute value may halt those students that previously intuitively understood how to solve equations and inequalities. E.g. $-2|x-3|+7=5$.
>
> After that, their intuition may break down when they come to the ε-δ proofs of limits in pre-calculus/calculus courses (this is my primary concern as it is one of the most important applications of absolute value in elementary mathematics).
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> For students that are fine through this point, if they stop there, then they made it through. If their goal was to get through class without learning the material, then good job! (I fell into this category my first time through the material. I eventually went back and retaught myself everything I skipped). There’s plenty more to do with absolute values and metrics in analysis and linear algebra courses after this point of

course, and the further you go, the more you are relying on an incomplete understanding.

Sure, if you're not planning on a career involving advanced mathematics, your intuition and memorization abilities may be able to get you through a few courses in algebra and possibly even a course in calculus, but all you will have accomplished is the ability to perform on demand and you will not have learned the mathematics we are trying to get you to learn. If your goal is to get through mathematics without learning mathematics, or to teach math to students without them learning math, then I am not sure why you are even reading this. It isn't for you.

So if you aren't teaching the mathematical definition of absolute value and how to use it, then you are further perpetuating the myth that you have to be a math person. You have to be able to intuit the answer without working with all of the tools.

This is absurd! The only way to know the answer is to know it? Even worse is to ask students to memorize a series of rules of the form, $|x| = k \Rightarrow x = k \text{ or } x = -k$, $|x| < k \Leftrightarrow -k < x < k$, and $|x| > k \Leftrightarrow x > k \text{ or } x < -k$.

Now you may think those rules are absurd and would never be taught to students, but they probably weren't presented to you that way. They are much more likely to be presented in the form of an example followed by, "so whenever you are in this situation, this is what you do" avoiding the use of mathematical language (which we avoid to a detrimental degree in favor of "plain-English" explanations anyway).

How $|x| < 2$ is usually taught:

"Let's see how we can solve something like $|x| < 2$. First, let's write it on the board:"

$$|x| < 2$$

"Next, in order to get rid of the absolute value, we write the compound sentence:"

$$-2 < x < 2$$

"We can see how this might work by drawing a number line and being able to see that this would clearly be the case."

"So whenever you have a situation with absolute value on one side, and a positive number on the other, you always go down and write the compound inequality with the opposite of the number on the left and the positive number on the right with what is inside the absolute value in-between."

This is even more subversive than at first appearance, because there is a good chance that many teachers presenting this material do not really know themselves why this is the case. If students ask prompting questions at this point, it may illustrate just how little the teacher knows, or the teacher may present information that is actually not true in an attempt to make this work for students.

So what's the problem with doing things case-by-case? Don't worry, I'm going to break it down for you. Before we start, let's enumerate the case-by-case rules:

(1). $|x| = k \Leftrightarrow x = k \text{ or } x = -k$,
(2). $|x| < k \Leftrightarrow -k < x < k$, and
(3). $|x| > k \Leftrightarrow x > k \text{ or } x < -k$.

First, one of the biggest issues with student comprehension is the idea that students cannot adapt to new situations, and forcing them to remember case-by-case situations further perpetuates that condition.

Second, the students aren't going to remember the rules (especially the last one which most texts seem to try to avoid).

And finally, because the rules are wrong. The first two rules assume $k \geq 0$ and the second disagrees with the others.

The second rule is a compound sentence, $x < k$ AND $x > -k$. The first and third rules are compound sentences $x > k$ OR $x < -k$. The fact that one is "and" and the other is "or" is not trivial.

So we have two concerns, how can we:

(1). Have an absolute value definition that is both correct AND useful, and
(2). Teach this effectively?

Depending on your class and how it is presented, some students may be able to reason out an algorithm to generate the absolute value of a number based on the type of input. This has educational value, especially the first few times it is presented in an algebra setting. But unless your class has been brought up in a strong math environment (either through the school system, curriculum, honors classes, or some individually fantastic teachers), I'm sorry to say that you're not usually going to be able to get this out of a class.

I usually ask students what the definition of absolute value is in advanced algebra, precalculus, calculus, and even calculus 2 courses and the answer I invariably get is "distance from zero" (and sometimes the aforementioned "positive of a number").

Now, if you can get your class to come up with the algorithm as part of a group or on their own, then you are in a great class or are a great teacher. That's the ideal for us at this point and it will make using the algorithm/definition so much easier for us and them in the future. There are some class activities you can do to help guide your students to the algorithm/definition (and I recommend them if time permits—but we all know time rarely permits). I include the outline of an activity below, but let us continue as though the algorithm needs to be presented.

Activity (Estimated Time: ~10 minutes)

Present students with a series of numbers: for instance, $3, -1, \frac{1}{2}, 0, -\pi, 17, 4, -1.2 \times 10^7$.

Note: A variety of numbers is nice, but try to keep it to no more than 4-6 unless they need further practice. At this point, we are assuming students already know how to find the absolute value of a number and are instead trying to coax an algebraic definition out of them. If they still need help with finding the absolute value of a number, this activity may be out of their reach.

Ask students to go through and find the absolute value of each number.

Next, ask students to describe the process they used for *each* number either in writing, with a partner, or as part of the class discussion.

We expect to get answers like,

"3 is already positive, so we left it alone."
"-1 is negative, so we *made it positive*."
"0 is not negative, so we didn't do anything to it."

If students do say something like, "made it positive", ask them to clarify or ask another student to help out by coming up with a math operation that will make a negative number positive. Also, try to guide them to using the term, "nonnegative" when describing the numbers that are left alone.

Then we ask students what process we need to go through when presented with a number of unknown value, x, to find its absolute value, $|x|$.

Depending on your class or on individual groups in your class, you may need to help them as far as asking leading questions like, "It seems like we do two different things depending on the value of x, how can we determine which we do and write it in mathematics?"

Eventually, the class should have a working definition of absolute value of a number, and it will hopefully be more solidified in their heads than just presenting it outright.

We turn to the language of mathematics to generate a useful definition for us, one that we can actually use to solve equations and inequalities:

$$|x| = \begin{cases} x, & if\ x \geq 0, \\ -x, & if\ x < 0 \end{cases}$$

We can elaborate on this definition for any unfamiliar with the piecewise notation:

If $x \geq 0$, then $|x| = x$ or if $x < 0$, then $|x| = -x$.

This is our “workhorse” that we will be using to solve any number of relations involving absolute value.

Now let’s solve each example presented in our original questions about absolute value using this definition/algorithm.

$$|x| = 2$$

If x is nonnegative, then x is 2; or if x is negative, then x is -2.

Thus, $x = 2$ or $x = -2$.

$$|x| < 2$$

If x is nonnegative, then $x < 2$. That is, $0 \leq x < 2$.

Also, if $x < 0$, then $-x < 2$. This tells us that $-2 < x < 0$.

We have $-2 < x < 0$ or $0 \leq x < 2$, which tells us that $-2 < x < 2$.

Note that even though this last sentence is translated as an “AND” sentence, the boundaries on the “OR” sentence in our work makes them equivalent.

Finally,

$$|x| > 2$$

If x is nonnegative, then $x > 2$. That is, if $x > 0, x > 2$. This is trivially true for all $x > 2$, so $x > 2$.

On the other hand, if x is negative, then $-x > 2$ or $x < -2$. This again is trivially true for all $x < -2$ so there is no needed extra boundary for this inequality.

Our final solution is $x > 2$ or $x < -2$.

If you are like others that are naturally good at mathematics, then you are probably infuriated by these examples. "There is so much unnecessary garbage here! Just give me the answer to the problem! Teach the shortcuts!"

I really do not believe that students should solve problems like this every time. That being said, students should be *able* to solve problems like this and presenting the information in this manner has true pedagogical worth.

Students should know that they can fall back on the rigor of mathematics when they come to a problem that they cannot solve. This trains students to be better at logic and reasoning and to understand that mathematics is not just a series of arcane symbols and secret processes that must be memorized in order to make sense of math. It teaches actual math, not the memorization of math someone else has done for them.

I always tell my students that I don't care if they get the correct answers unless they used an appropriate technique. If they were the first people to figure out that $2 + 3 = 5$, then we can talk about the importance they got the number correct. If they were out in the real world doing arithmetic, then yes, accuracy is important. But mathematics is more than arithmetic, and it is definitely more than getting the right answer. It is problem solving and applying techniques to situations you did not know they could be applied to. Presenting only the heuristic definition of a mathematical object and expecting students to intuitively use it to solve problems is tantamount to giving up on mathematics as a subject entirely.

There is nothing wrong with teaching shortcut techniques so long as the shortcut techniques are not presented in a vacuum. The strength of mathematics comes from its rigor. Anytime students feel uneasy or unsure of how to proceed, it must not be seen as a failure in their intuition or in them. How can we expect students to intuit things that took centuries to develop? It is instead a failure in the foundation of the mathematics they know and that we taught them.

Everything in mathematics makes sense if you teach mathematics rigorously. It's when teachers begin to handwave or expect students to "just understand" something that a student's foundation begins to become shaky. The more the student balances on that shaky foundation, the more anxiety they feel towards mathematics and the more likely they are to misunderstand, rely on memorization, or give up on understanding mathematics entirely (or any combination thereof).

I can appreciate that rigor is more difficult to teach than the heuristic definition and processes, but the heuristic definition should always be considered supplementary. The heuristic definition is the tool we use to develop understanding of the rigorous definition.

We should also make sure not to give students too much at once of course. I absolutely would present the true definition of absolute value to students learning absolute value for the first time, but I would not use mathematical notation. This is because students usually learn absolute value before they learn algebra. But we can still teach them the undoubtedly worthwhile definition/algorithm: “To find the absolute value of a number: if it is nonnegative, leave it alone; if it is negative, take its opposite,” and then guide them to the algebraic definition once they have the language to describe it algebraically.

It doesn’t help students to hide the truth from them, and if it makes things make more sense, why not share that information with them? It is up to you as teachers and students to seek that information out and to relate the truth of it in ways that make sense without obscuration.